Olimpiada Naţională de Matematică
Etapa locală - 15.02.2014
Clasa a XI-a M_1

Soluţii şi bareme

1. a) Dezvoltând determinantul obţinem: $f(x)=e^{2x^2+2x+2a}+2e^{-x^2-x-a}-3=e^{2\left(x^2+x+a\right)}+\dfrac{2}{e^{\left(x^2+x+a\right)}}-3$ ……2p

Notând : $e^{\left(x^2+x+a\right)}=t>0$, ecuaţia $f(x)=0$ devine :

$t^2+\dfrac{2}{t}-3=0\Leftrightarrow\dfrac{t^3-3t+2}{t}=0\Leftrightarrow\dfrac{(t-1)^2(t+2)}{t}=0\Leftrightarrow(t-1)^2(t+2)=0\Rightarrow t_1=1>0;t_2=-2<0$ 1p

Revenind la notaţie, vom avea : $e^{\left(x^2+x+a\right)}=1\Rightarrow x^2+x+a=0(*)$.

Ecuaţia are soluţii reale $\Leftrightarrow\Delta\geq 0\Leftrightarrow 1-4a\geq 0\Leftrightarrow a\in\left(-\infty,\dfrac{1}{4}\right]$. 1p

a) Ecuaţia $f(x)=0$ are toate soluţiile strict negative $\Leftrightarrow$ ecuaţia $(*)$ are soluţiile x_1,x_2 strict negative.....1p

$\Leftrightarrow\begin{cases}\Delta\geq 0\\ S<0 \text{ unde } S=x_1+x_2=-1; P=x_1\cdot x_2=a\\ P>0\end{cases}\Rightarrow$ 1p

$\begin{cases}a\in\left(-\infty,\dfrac{1}{4}\right]\\ -1<0 \quad \text{adevarat}\\ a\in(0,\infty)\end{cases}\Leftrightarrow a\in\left(0,\dfrac{1}{4}\right]$.

. .....1p

2. 1) $\Rightarrow$ **2)** $A+B+B^2+\ldots+B^{k-1}=O_n\,|\cdot B\Rightarrow AB+B^2+\ldots+B^{k-1}+B^k=O_n\overset{AB=A+B}{\Rightarrow}\underbrace{A+B+B^2+\ldots+B^{k-1}}_{O_n}+B^k=O_n\Rightarrow B^k=O_n$ …..3p

2) $\Rightarrow$ **1)** Ştim că $B^k=O_n,\forall k\in\mathbb{N},k\geq 2$ .Demonstrăm prin metoda inducţiei matematice afirmaţia de la (1) 1p

$P(k):A+B+B^2+\ldots+B^{k-1}=O_n,\forall k\in\mathbb{N},k\geq 2$

$I)\,P(2):A+B=O_n$ *de verificat*

$O_n=B^2=AB^2=(AB)B\overset{AB=A+B}{=}(A+B)B=AB+B^2\overset{B^2=O_n}{=}AB=A+B\Rightarrow A+B=O_n\Rightarrow P(2)$ adevărat. 1p

$II) P(k) \to P(k+1)$

$P(k): A+B+B^2+...+B^{k-1}=O_n$, pentru $k \in \mathbb{N}, k \geq 2$ fixat , ipoteză de inducţie

$P(k+1): A+B+B^2+...+B^{k-1}+B^k=O_n$, trebuie demonstrat

Din $P(k) \Rightarrow A+B+...+B^{k-1}=O_n \Big| +B^k \left(=O_n\right) \Rightarrow A+B+B^2+...+B^{k-1}+B^k=O_n \Rightarrow P(k+1): A+B+B^2+...+B^{k-1}+B^k=O_n$ adevărat. 2p

3. $\displaystyle\sum_{k=1}^{n}\frac{k^2-k+1}{k^4+k}=\sum_{k=1}^{n}\frac{k^2-k+1}{\left(k^4+k\right)}=\sum_{k=1}^{n}\frac{k^2-k+1}{k\left(k^3+1\right)}=\sum_{k=1}^{n}\frac{\cancel{k^2-k+1}}{k\left(k+1\right)\cancel{\left(k^2-k+1\right)}}=$ 2p

$\displaystyle=\sum_{k=1}^{n}\frac{1}{k\left(k+1\right)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1-\frac{1}{n+1}=\frac{n}{n+1}$ 2p

$\displaystyle\Rightarrow\lim_{n\to\infty}\left(\sqrt[n]{\sum_{k=1}^{n}\frac{k^2-k+1}{k^4+k}}\right)^{n^2+1}=\lim_{n\to\infty}\left(\sqrt[n]{\frac{n}{n+1}}\right)^{n^2+1}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{\frac{n^2+1}{n}}=\lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-\left(n^2+1\right)}{n(n+1)}}=e^{-1}=\frac{1}{e}$ 3p

4. $\displaystyle\lim_{x\to\infty}x^{\sim}\left(\sqrt{x+1}-2\sqrt{x}+\sqrt{x-1}\right)=\lim_{x\to\infty}x^{\sim}\left(\sqrt{x+1}-\sqrt{x}-\sqrt{x}+\sqrt{x-1}\right)=\lim_{x\to\infty}x^{\sim}\left(\frac{\cancel{x}+1-\cancel{x}}{\sqrt{x+1}+\sqrt{x}}+\frac{\cancel{x}-1-\cancel{x}}{\sqrt{x-1}+\sqrt{x}}\right)=$ 2p

$\displaystyle=\lim_{x\to\infty}x^{\sim}\left(\frac{1}{\sqrt{x+1}+\sqrt{x}}+\frac{-1}{\sqrt{x-1}+\sqrt{x}}\right)=\lim_{x\to\infty}x^{\sim}\left(\frac{\sqrt{x-1}+\cancel{\sqrt{x}}-\sqrt{x+1}-\cancel{\sqrt{x}}}{\left(\sqrt{x+1}+\sqrt{x}\right)\left(\sqrt{x-1}+\sqrt{x}\right)}\right)=$ 1p

$\displaystyle=\lim_{x\to\infty}x^{\sim}\left(\frac{\cancel{x}-1-\cancel{x}-1}{\left(\sqrt{x+1}+\sqrt{x}\right)\left(\sqrt{x-1}+\sqrt{x}\right)\left(\sqrt{x-1}+\sqrt{x+1}\right)}\right)=$ 1p

$\displaystyle=\lim_{x\to\infty}\frac{-2x^{\sim}}{x^{\frac{3}{2}}\left(\sqrt{1+\frac{1}{x}}+1\right)\left(\sqrt{1-\frac{1}{x}}+1\right)\left(\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{1}{x}}\right)}\overset{not}{=}l$ 2p

Discuţie:

1) Pentru $\sim<\frac{3}{2}\Rightarrow l=0$ 2) Pentru $\sim=\frac{3}{2}\Rightarrow l=-\frac{1}{4}$ 3) Pentru $\sim>\frac{3}{2}\Rightarrow l=-\infty$ 1p

Olimpiada Naţională de Matematică

Etapa locală - 15.02.2014

Clasa a XI-a M_2

Soluţii şi bareme

1. $D=\mathbb{R}\Rightarrow$ nu există asimptote verticale. Cum $\lim\limits_{x\to\pm\infty} f(x)=\infty\Rightarrow$ nu există asimptote orizontale . ……2p

Studiem existenţa asimptotei oblice la $-\infty$:

$$m=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{x}=\lim_{x\to-\infty}\frac{|x|\sqrt{1+\frac{2}{x^2}}}{x}=\lim_{x\to-\infty}\frac{-x\sqrt{1+\frac{2}{x^2}}}{x}=-1\ ,$$ …….1p

$$n=\lim_{x\to-\infty}\left[f(x)+x\right]=\lim_{x\to-\infty}\left[\underbrace{\sqrt{x^2+2}}_{\downarrow\infty}+\underbrace{x}_{\downarrow-\infty}\right]=\lim_{x\to-\infty}\frac{x^2+2-x^2}{\sqrt{x^2+2}-x}=\frac{2}{\infty}=0\ ,$$ ……1p

deci $y=-x$ asimptotă oblică la $-\infty$. ……0,5p

Studiem existenţa asimptotei oblice la $+\infty$:

$$m=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{x}=\lim_{x\to\infty}\frac{|x|\sqrt{1+\frac{2}{x^2}}}{x}=\lim_{x\to\infty}\frac{x\sqrt{1+\frac{2}{x^2}}}{x}=1,$$ …….1p

$$n=\lim_{x\to\infty}\left[f(x)-x\right]=\lim_{x\to\infty}\left[\underbrace{\sqrt{x^2+2}}_{\downarrow\infty}-\underbrace{x}_{\downarrow\infty}\right]=\lim_{x\to-\infty}\frac{x^2+2-x^2}{\sqrt{x^2+2}+x}=\frac{2}{\infty}=0\,,$$ …….1p

deci $y=x$ asimptotă oblică la $+\infty$. ……0,5p

2. $\begin{vmatrix}x&1&2\\1&2&x\\2&x&1\end{vmatrix}\overset{L_1\to L_1+L_2+L_3}{=}\begin{vmatrix}x+3&x+3&x+3\\1&2&x\\2&x&1\end{vmatrix}=(x+3)\cdot\begin{vmatrix}1&1&1\\1&2&x\\2&x&1\end{vmatrix}=(x+3)\cdot(2+x+2x-4-x^2-1)==(x+3)\cdot(-x^2+3x-3)$ ……4p

Cum determinantul este nul $\Leftrightarrow\ (x+3)\cdot(-x^2+3x-3)=0$, adică $x+3=0\Leftrightarrow x=-3$ ..…1p

sau $-x^2+3x-3=0$, dar $\Delta=-3<0$ şi ecuaţia nu are soluţii reale. Deci $S=\{-3\}$. ……2p

3. $\lim\limits_{x\to 1}\dfrac{\sin\left(x^2-1\right)}{\operatorname{tg}\left(x^3-1\right)}\cdot\dfrac{e^{x^2-1}}{x^4-1}=\lim\limits_{x\to 1}\dfrac{\overbrace{\dfrac{\sin\left(x^2-1\right)}{x^2-1}}^{\nearrow 1}\cdot\cancel{(x-1)}(x+1)}{\underbrace{\dfrac{\operatorname{tg}\left(x^3-1\right)}{x^3-1}}_{\searrow 1}\cdot\cancel{(x-1)}\left(x^2+x+1\right)}\cdot\dfrac{\overbrace{\dfrac{e^{x^2-1}-1}{x^2-1}}^{\nearrow 1}\cdot\cancel{\left(x^2-1\right)}}{\cancel{\left(x^2-1\right)}\left(x^2+1\right)}=\dfrac{2}{3}\cdot\dfrac{1}{2}=\dfrac{1}{3}.$ ……..7p

4. a) $\det\left(A+3I_3\right)=\begin{vmatrix}4&0&-1\\0&3&0\\1&0&4\end{vmatrix}=48+3=51.$ …….2p

b) $A^2=\begin{pmatrix}0&0&-2\\0&0&0\\2&0&0\end{pmatrix}$, $A^3=A^2\cdot A=\begin{pmatrix}-2&0&-2\\0&0&0\\2&0&-2\end{pmatrix}$, $A^5=A^3\cdot A^2=\begin{pmatrix}-4&0&4\\0&0&0\\-4&0&-4\end{pmatrix}$ şi …….1p

$A^5+4A=\begin{pmatrix}-4&0&4\\0&0&0\\-4&0&-4\end{pmatrix}+\begin{pmatrix}4&0&-4\\0&0&0\\4&0&4\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=O_3.$ …..1p

c) Din (b) $\Rightarrow A^5+4A=O_3\Rightarrow A^5=-4A$, $A^9=A^5\cdot A^4=-4A^5=(-4)^2A$, $A^{13}=A^9\cdot A^4=(-4)^2A^5=(-4)^3A$,…, $A^{2013}=(-4)^{503}A,\Rightarrow$ …..1p

$A+A^5+A^9+\ldots+A^{2013}=A+(-4)A+(-4)^2A+\ldots+(-4)^{503}A=\left[1+(-4)+(-4)^2+\ldots+(-4)^{503}\right]A=$ ……1p

$=\dfrac{(-4)^{504}-1}{-4-1}A=\dfrac{1-4^{504}}{5}A.$ ……1p